Acyclic and Star Colorings of Joins of Graphs and an Algorithm for Cographs

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Abstract

An *acyclic* (resp. *star*) *coloring* of a graph is a proper vertex coloring such that the subgraph induced by every pair of colors is a disjoint collection of trees (resp. stars). In this paper, we consider the acyclic and star chromatic numbers of graphs that are decomposable with respect to the join operation, which builds a new graph from a collection of two or more disjoint graphs by adding all possible edges between them. In particular, we present a recursive formula for the acyclic chromatic number of joins of graphs and show that a similar formula holds for the star chromatic number.

The cographs have the unique property that they are recursively decomposable with respect to the join and disjoint union operations. We show that our results imply a linear time algorithm for finding optimal acyclic and star colorings of cographs.

1 Introduction

A *proper vertex coloring* (or *coloring*) of a graph $G = (V, E)$ is an assignment of colors to the vertices such that adjacent vertices receive distinct colors. The chromatic number of G, denoted $\chi(G)$, is the minimum number of colors required in any coloring of G. An *acyclic coloring* of a graph is a coloring such that the subgraph induced by the union of any two colors is a disjoint collection of trees. A *star coloring* of a graph is a coloring such that the subgraph induced by the union of any two colors is a disjoint collection of stars. The acyclic and star chromatic numbers of G are defined analogously to the chromatic number and are denoted by $\chi_a(G)$ and $\chi_s(G)$, respectively. Since a disjoint collection of stars constitutes a forest, every star coloring of a graph G is also an acyclic coloring and $\chi_a(G) \leq \chi_s(G)$. We will find it useful to consider the alternative definitions that result from the following observation.

Observation 1. *The following hold for any graph G :*

(*i*) *A coloring of G is an acyclic coloring if and only if every cycle in G uses at least three colors.*

(*ii*) *A coloring of G is a star coloring if and only if every path on four vertices in G uses at least three colors.*

The notions of acyclic and star coloring were introduced in 1973 (the latter by a different name) by Grünbaum [9], who studied them in the context of planar graphs.

Additionally, a number of results exist for acyclic colorings of graphs formed by certain graph operations. Results have been obtained for grids (which are the Cartesian products of paths) [5], as well as Cartesian products of trees [13], cycles [11], and complete graphs [12]. In Section 3, we describe the acyclic and star chromatic number of graphs formed by the join operation [10].

In Section 4, we turn our attention to algorithmic properties of acyclic and star colorings.

The study of these problems from an algorithmic point of view is motivated in part by their applications in the field of combinatorial scientific computing, where they model the optimal evaluation of sparse Hessian

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matrices. The general idea of the use of coloring in computing derivative matrices is the identification of entities which are essentially independent and thus may be computed concurrently; see [6] for a survey.

In fact, both acyclic and star colorings were discovered independently by the scientific computing community. The problem of finding an acyclic coloring that uses a minimum number of colors was shown to be **NP**-hard in [2], where it is called the "cyclic" coloring problem. It has also been shown that finding an optimal star coloring is **NP**-hard [3]. Both results hold even for bipartite graphs. Inapproximability results for both problems are given in [8].

Recently (also in the context of computing Hessian matrices), it was shown in [7] that every coloring of a chordal graph is also an acyclic coloring. Since recognizing and optimally coloring chordal graphs can be done in linear time, this result immediately implies a linear time algorithm for acyclic coloring problem on chordal graphs. A generalization of this result and other related results can be found in [14]. In particular, it is shown that cographs can be characterized in the following way.

Theorem 1.1 ([1, 14]). *Let G be a graph. Then the following are equivalent:*

(*i*) *G is a cograph;*

(*ii*) *G has no induced P_4;*

(*iii*) *every acyclic coloring of G is also star coloring.*

Thus, the cographs are interesting for reasons other than the nice decomposition properties that they exhibit. This well-studied class has many other characterizations; see [1, Theorem 11.3.3] for a partial list. We demonstrate that the results given in Section 3 can be used to develop a linear time algorithm for finding an optimal acyclic coloring of a cograph. Additionally, we show that the coloring obtained is also an optimal star coloring, as suggested by Theorem 1.1.

2 Preliminaries

In this section, we introduce some definitions and notation, as well as some prove some elementary results that will be useful in Sections 3 and 4. Throughout this paper, $\mathcal{I}$ will denote a finite index set. The *disjoint union* of a collection $\{G_i = (V_i, E_i)\}_{i\in\mathcal{I}}$ of pairwise disjoint graphs is the graph $G = (V, E)$, where $V = \bigcup_{i\in\mathcal{I}} V_i$ and $E = \bigcup_{i\in\mathcal{I}} E_i$.

Proposition 2.1. *The following hold for any graph $G = G_1 \bigcup G_2$:*

(*i*) $\chi(G) = \max\{\chi(G_1), \chi(G_2)\}$;

(*ii*) $\chi_a(G) = \max\{\chi_a(G_1), \chi_a(G_2)\}$;

(*iii*) $\chi_s(G) = \max\{\chi_s(G_1), \chi_s(G_2)\}$.

Proof. The proof follows from the simple observation that the graph with the lower chromatic number can be colored with some subset of the colors used by the other graph. □

The *join* of a collection $\{G_i = (V_i, E_i)\}_{i\in\mathcal{I}}$ of pairwise disjoint graphs is the graph $G = (V, E)$, where $V = \bigcup_{i\in\mathcal{I}} V_i$ and $E = \{ab \mid ab \in E_i, i \in \mathcal{I}\} \cup \{ab \mid a \in V_i, b \in V_j, i, j \in \mathcal{I}, i \neq j\}$.

Proposition 2.2. *The following hold for any graph $G = G_1 \bigoplus G_2$:*

(*i*) $\chi(G) = \chi(G_1) + \chi(G_2)$;

(*ii*) $\chi_a(G) \geq \chi_a(G_1) + \chi_a(G_2)$;

(*iii*) $\chi_s(G) \geq \chi_s(G_1) + \chi_s(G_2)$.

Proof. We first observe that G_1 and G_2 are induced subgraphs of G, and thus they must both be colored appropriately for any coloring of G. The proof then follows from the fact that every vertex in V_1 is adjacent to every vertex in V_2, which means that no color can occur on a vertex in V_1 and a vertex in V_2 simultaneously. □

We will write $G = V_1 \bigoplus V_2$ when $G = (V, E)$ is the graph that results from taking the join of two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ such that $V_1 \cap V_2 = \emptyset$. We denote by $|\phi|$ the number of colors used by a coloring $\phi : V \to \{1, \ldots, |\phi|\}$ of a graph $G = (V, E)$. Let ϕ be a coloring of a graph G such that G is either the disjoint union or join of a collection $\{G_i\}_{i \in \mathcal{I}}$ of graphs. We denote by ϕ_i the coloring of G_i obtained by restricting ϕ to V_i, where $|\phi_i|$ is the number of colors used by ϕ_i. The following proposition holds even when ϕ is an acyclic coloring or a star coloring.

Proposition 2.3. *Let ϕ be a coloring of a graph $G = G_1 \bigoplus G_2$. Then ϕ_1 and ϕ_2 are disjoint, and $|\phi| = |\phi_1| + |\phi_2|$. Moreover, the same is true when ϕ is an acyclic or star coloring of G.*

Proof. This follows immediately from Proposition 2.2. □

3 Main Results

For ease of exposition, we will focus on the case where G is the join of exactly two graphs. We will then demonstrate that these results generalize to joins of arbitrarily large collections of graphs.

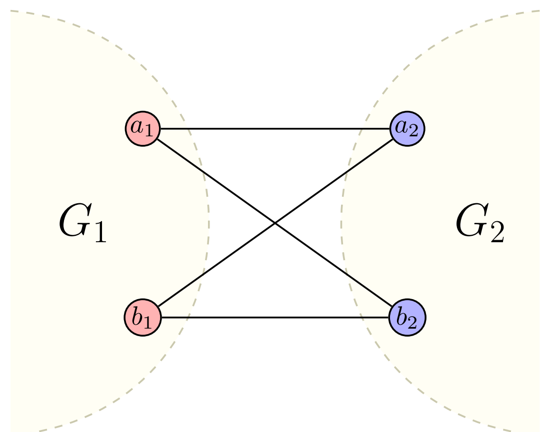

Figure 1: Illustration of the proof of Theorem 3.1. If $|\phi_1| < |V_1|$ and $|\phi_2| < |V_2|$, then there exist vertices $a_1, b_1 \in V_1$ and $a_2, b_2 \in V_2$ that form a bichromatic C_4.

We are now ready to present one of the main theorems, which relates $\chi_a(G)$ to $\chi_a(G_1)$ and $\chi_a(G_2)$.

Theorem 3.1. *Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be graphs. Then*

$$\chi_a(G_1 \bigoplus G_2) = \chi_a(G_1) + \chi_a(G_2) + \min\{|V_1| - \chi_a(G_1), |V_2| - \chi_a(G_2)\}.$$

Proof. $(\leq)$: We prove this direction by presenting an algorithm that, given optimal acyclic colorings of G_1 and G_2, produces an acyclic coloring ϕ of G that uses the desired number of colors. Let ϕ_1 and ϕ_2 be arbitrary optimal acyclic colorings of G_1 and G_2, respectively, where, as follows from Proposition 2.3, ϕ_1 and ϕ_2 are disjoint. Assume without loss of generality that $|V_1| - \chi_a(G_1) \leq |V_2| - \chi_a(G_2)$. We construct ϕ as follows.

- Color those vertices in $V_2 \subseteq V$ the same as they are colored by ϕ_2.
- Color those vertices in $V_1 \subseteq V$ with a new coloring ϕ_1' such that each $v \in V_1$ receives a unique color and ϕ_1' is disjoint from ϕ_1 and ϕ_2.

To see that this process results in $|\phi|$ having the desired size, observe that the difference between $|\phi_1|$ and $|\phi_1'|$ is exactly $|V_1| - \chi_a(G_1)$, which was assumed without loss of generality to be no greater than $|V_2| - \chi_a(G_2)$. Thus we have demonstrated a method for constructing the desired coloring ϕ, which completes this direction of the proof.

$(\geq)$: Let ϕ be an optimal acyclic coloring of G, and assume for the sake of contradiction that

$$\chi_a(G) < \chi_a(G_1) + \chi_a(G_2) + \min\{|V_1| - \chi_a(G_1), |V_2| - \chi_a(G_2)\}.$$

It follows that

$$\chi_a(G) < \chi_a(G_1) + \chi_a(G_2) + |V_1| - \chi_a(G_1)$$

and

$$\chi_a(G) < \chi_a(G_1) + \chi_a(G_2) + |V_2| - \chi_a(G_2),$$

which can be combined with Proposition 2.3, along with the fact that ϕ is an acyclic coloring of G, to obtain

$$|\phi_1| + |\phi_2| < \chi_a(G_2) + |V_1| \tag{1}$$

and

$$|\phi_1| + |\phi_2| < \chi_a(G_1) + |V_2|. \tag{2}$$

We will now show that this implies that ϕ uses few enough colors that we must have $|\phi_1| < |V_1|$ and $|\phi_2| < |V_2|$, which will lead to a contradiction. Note that Proposition 2.3 also implies that $\chi_a(G_1) \leq |\phi_1|$ and $\chi_a(G_2) \leq |\phi_2|$, and note also that all quantities in (1) and (2) are nonnegative. Thus we may subtract $|\phi_2|$ from the left-hand side and $\chi_a(G_2)$ from the right-hand side of (1), and we may likewise subtract $|\phi_1|$ from the left-hand side and $\chi_a(G_1)$ from the right-hand side of (2). In doing so, we show that ϕ_1 must use fewer than $|V_1|$ colors and ϕ_2 must use fewer than $|V_2|$ colors, as desired. Consequently, there must exist vertices $a_1, b_1 \in V_1$ and $a_2, b_2 \in V_2$ such that $\phi_1(a_1) = \phi_1(b_1)$ and $\phi_2(a_2) = \phi_2(b_2)$. It follows that the vertices in $\{a_1, a_2, b_1, b_2\}$ form a bichromatic C_4 in G (depicted in Figure 1) which contradicts the fact that ϕ is an acyclic coloring of G, and thus the proof is complete. □

We now develop an analogous theorem for star coloring.

Theorem 3.2. *Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be graphs. Then*

$$\chi_s(G_1 \bigoplus G_2) = \chi_s(G_1) + \chi_s(G_2) + \min\{|V_1| - \chi_s(G_1), |V_2| - \chi_s(G_2)\}.$$

Proof. The proof follows from the observation that any bichromatic cycle appearing in the proof of Theorem 3.1 implies a bichromatic P_4.

□

Corollary 3.3. *Let $\{G_i = (V_i, E_i)\}_{i \in \mathcal{I}}$ be a finite collection of graphs. Then*

(i) $\chi_a\left(\bigoplus_{i\in\mathcal{I}} G_i\right) = \sum_{i\in\mathcal{I}} \chi_a(G_i) + \min_{j\in\mathcal{I}} \left\{ \sum_{i\in\mathcal{I}, i\neq j} (|V_i| - \chi_a(G_i)) \right\};$

(ii) $\chi_s\left(\bigoplus_{i\in\mathcal{I}} G_i\right) = \sum_{i\in\mathcal{I}} \chi_s(G_i) + \min_{j\in\mathcal{I}} \left\{ \sum_{i\in\mathcal{I}, i\neq j} (|V_i| - \chi_s(G_i)) \right\}.$

Proof. Observing that the join operation is commutative and associative, we obtain the result by using induction on $|\mathcal{I}|$. □

4 Cographs

In this section, we present a linear time algorithm for finding optimal acyclic and star colorings of cographs. Our algorithm works on the cotree (defined below) in a way that is typical for algorithms on cographs.

We select the following definition, which is one of many equivalent definitions of the class of cographs, as it will be most useful for our purposes.

Definition 1 (cograph)**.** *A graph $G = (V, E)$ is a cograph if and only if one of the following is true:*

(i) $|V| = 1$;

(ii) *there exists a collection* $\{G_i\}_{i\in\mathcal{I}}$ *of cographs such that* $G = \bigcup_{i\in\mathcal{I}} G_i$;

(iii) *there exists a collection* $\{G_i\}_{i\in\mathcal{I}}$ *of cographs such that* $G = \bigoplus_{i\in\mathcal{I}} G_i$.

Cographs can be recognized in linear time [4], where most recognition algorithms also produce a special decomposition structure when the input graph G is a cograph. We now introduce this structure, which is often used in algorithms designed to work on cographs. We associate with a cograph G a tree T_G called a *cotree*, whose leaves correspond to the vertices of G and whose internal nodes are labeled either 0 or 1, corresponding to the disjoint union and join operations, respectively, in the following way. Let $t \in T$ be an internal node with children $\{t_i\}_{i\in\mathcal{I}}$. If t is a 0-node, then t corresponds to the disjoint union of $\{t_i\}_{i\in\mathcal{I}}$, Otherwise, t corresponds to the join of $\{t_i\}_{i\in\mathcal{I}}$. Every node in T_G has as descendants some subset $A \subseteq V$ of the vertices of G. Therefore, it is natural to identify each node in the tree with the graph induced in G by this set of vertices, which we denote by t_A. In this way, the cotree describes a decomposition of G such that the root of T_G corresponds to G itself. It follows from the definition of the cotree that two vertices in G are adjacent if and only if their lowest common ancestor in T_G is a 1-node.

The *canonical* cotree T_G is unique and has the property that every path from a leaf to the root alternates between 0-nodes and 1-nodes. G can also be represented by one or more cotrees which do not necessarily possess this property, but whose internal nodes all have exactly two children. Such *binary* cotrees can be easier to work with algorithmically, as is the case in the proof of the following theorem.

Theorem 4.1. *An optimal acyclic coloring of a cograph can be found in linear time. Furthermore, the obtained coloring is also a star coloring.*

Proof. We assume that a cograph G has been given along with a cotree T_G, which can be assumed without loss of generality to be binary. We initialize $\chi_a(t) = 1$ for every leaf $t \in T_G$ and initialize the coloring that will be constructed by assigning each vertex the same color. The algorithm proceeds by traversing the cotree starting with the leaves, such that no node is visited before both of its children have been visited. When we visit a node $t \in T_G$ with children t_1 and t_2, we do the following.

- If t is a 0-node, we use the process described in the proof of Proposition 2.1 to obtain a coloring that uses $\chi_a(t) = \max\{\chi_a(t_1), \chi_a(t_2)\}$ colors.
- If t is a 1-node, we use the process described in the proof of Theorem 3.1 to obtain a coloring that uses $\chi_a(t) = \chi_a(t_1) + \chi_a(t_2) + \min\{|V_{t_1}| - \chi_a(t_1), |V_{t_2}| - \chi_a(t_2)\}$ colors.

Clearly, any coloring produced by the algorithm will be acyclic, as we have already shown that the coloring method applied at each 1-node and 0-node in T_G produces an acyclic coloring. Furthermore, it follows from Theorem 3.1 and Proposition 2.1 that the coloring produced at each $t_A \in T$ is an optimal coloring for the subgraph induced by $A \subseteq V$ in G, which includes the root of T_G. As the root corresponds to G, and our algorithm clearly runs in linear time, we have demonstrated the desired algorithm for acyclic coloring. That the coloring obtained is also an optimal star coloring follows from Theorem 1.1. □

Example. We demonstrate the behavior of the algorithm on the example shown in Figure 2. In particular, while we could use any cotree for G, we will use the binary cotree shown in Figure 2(c). We begin with

$$\chi_a(t_{\{a\}}) = \chi_a(t_{\{b\}}) = \cdots = \chi_a(t_{\{h\}}) = 1.$$

Now moving up the tree, we get $\chi_a(t_{\{a,b\}}) = \chi_a(t_{\{c,d\}}) = \max\{1, 1\} = 1$ and $\chi_a(t_{\{e,f\}}) = \chi_a(t_{\{g,h\}}) = \chi_a(t_{\{e,f,g,h\}}) = 2$. Next, we compute $\chi_a(t_{\{a,b,c,d\}}) = 1 + 1 + \max\{1-1, 1-1\} = 2$.

Finally, the most interesting case is the root of T_G, which corresponds to G itself. In particular, we have $G = G_1 \bigoplus G_2$, where $G_1 = t_{\{a,b,c,d\}}$ and $G_2 = t_{\{e,f,g,h\}}$. Thus $\chi_a(G_1) = 3, \chi_a(G_2) = 2$, and $|V_1| = |V_2| = 4$. Therefore, $\chi_a(G) = \chi_a(t_{\{a,b,c,d,e,f,g,h\}}) = 3 + 2 + \min\{4-3, 4-2\} = 3 + 2 + 1 = 6$.

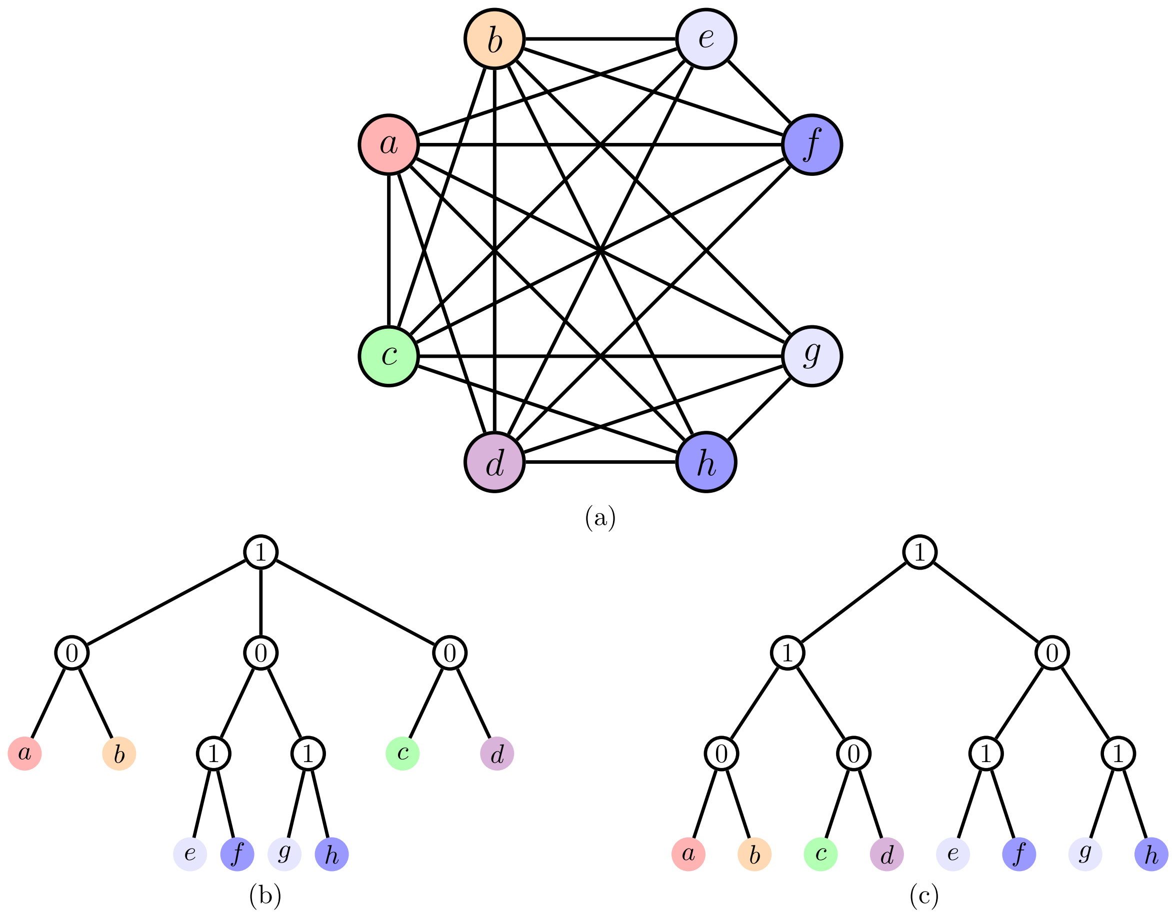

Figure 2: (a) A cograph G; (b) its canonical cotree T_G; (c) a binary cotree T. The graph is shown along with an optimal acyclic coloring, which is (necessarily) also an optimal star coloring.

5 Concluding Remarks

We have shown that the acyclic and star chromatic numbers of graphs formed by the join operation can be expressed recursively in terms of the graphs that they compose. We have also demonstrated some algorithmic properties of these problems with respect to the join operation. Our results, along with the recursive properties of the cographs, yield a linear time algorithm for this class.

We hope that the results presented here will lead to efficient algorithms for the efficient computation of Hessian matrices that exhibit special structure. We also hope that these results can be generalized. A first approach would be to attempt to apply other decompositions whose algorithmic properties are well known, as well as the classes that have special structure with respect to them. Possibilities include modular, split, tree, and 2-join decompositions.

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